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A road map to higher genus Gromov-Witten invariants of Calabi-Yau quintics*

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Abstract: This is a survey of using NMSP method to study higher genus Gromov-Witten invariants of Calabi-Yau quintics. It emphasizes on how and why the various methods are introduced to solve several important conjectures for higher genus Gromov-Witten invariants of Calabi-Yau quintics.

Key words: Gromov-Witten invariants; Calabi-Yau manifolds

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1　Enumerative geometry of counting curves

Enumerative geometry is a research area of algebraic geometry where we count the number of geometric objects. The simplest example is to count the number of lines in the plane passing through two given points. Apparently, the answer is 1, i. e. , there is only one line passing through two given points.

The first golden age of enumerative geometry in modern time is near the end of the nineteenth century. One of the key figures among many enumerative geometers is Hermann Schubert. Many sophisticated methods were developed to solve various enumerative geometric problems. The whole subject is sometimes called Schubert Calculus. Its modern treatment can be found in the book Intersection Theory by Fulton(1984).

We can look at a simple example.

Example 1 Number of conics in $\mathbb{P}^2$ passing through 5 general points.

Consider a conic in the projective plane $\mathbb{P}^2$. The conic $\mathcal{C}$ is the zero locus of a degree two homogeneous polynomial

$$f\left(x, y, z\right) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5yz + a_6xz = 0.$$

Hence the set of conics can be parametrized by $\mathbb{P}^5$ such that the conic $\mathcal{C}$ corresponds to the point $\left[a_1,\cdots,a_6\right] \in \mathbb{P}^5$, where $\{a_i\}$ are the coefficients of the the polynomial f. The set of conics passing through a given point is a linear condition on the coefficients of $f\left(x, y, z\right)$, hence it is a linear hypersurface of $\mathbb{P}^5$. The number of conics passing through general 5 points equals the intersection number of 5 linear subspaces, which is 1.

This simple example illustrates the standard way to do enumerative geometry. The geometric objects (conics in the example) form a set ($\mathbb{P}^5$ in the example), called moduli space. A condition on the objects (conics containing a given point in the example) is a subset, usually a divisor, in the moduli space (a linear hypersurface in the example). The enumerative number will be the intersection number of these divisors. How many divisors we

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need depends on the dimension of the moduli space.

Example 2 Number of conics in $\mathbb{P}^2$ tangent to 5 general lines.

Conics tangent to a given line L is a quadratic condition. Hence the set of conics tangent to a given line is a quadric Q_L in $\mathbb{P}^5$. The number of conics tangent to 5 general lines would be the number of the intersection of 5 quadrics, which equals $2^5 = 32$. In fact, the answer is 1. What goes wrong?

A line is tangent to a conic if the multiplicity of the intersection of the line with the conic is bigger than 1. Conics can be degenerate into a double line given by an equation

$$(ax + by + cz)^2 = 0.$$

Any line intersects any degenerate conic with multiplicity two, thus the set S of double lines is contained in any divisor Q_L. Even though the five lines L_i are in general position, the corresponding divisors Q_{L_i} don't intersect properly. This is an excess intersection. The intersection theory has a method to deal with this issue. Using it, we obtain the actual number of conics tangent to five general lines is 1.

In summary, many enumerative geometric problems can be reformulated into a problem of intersection theory on the moduli space of geometric objects to be enumerated. And in many cases, the intersection is an excess intersection and hence cannot be naively computed.

2 Enumerating rational curves on Calabi-Yau quintics

A Calabi-Yau quintic X is the zero locus of a degree 5 homogeneous polynomial in $\mathbb{P}^4$, e. g. ,

$$X = \{ x_1^5 + \cdots + x_5^5 = 0 \} \subset \mathbb{P}^4.$$

Clemens proposed a conjecture that there are only finitely many rational curves of given degree on a general Calabi-Yau quintic X. The conjecture has been proved for low degrees. The next question is the number of rational curves of a given degree on X. For low degrees, the number of rational curves has been computed (Katz, 1983;Katz,1986). For example, the number of lines on a general quintic is 2 875, and the number of conics on a general quintic is 609 250.

In the seminal paper by physicists Candelas et al. (1991), the number of rational curves of all degrees on quintics X is calculated using a physical conjectural technique, called mirror symmetry.

Let $N_{g,d}$ represent the number of genus g degree d curves on X. Then we have a generating function

$$F_g(q) = \sum_{d=0}^{\infty} N_{g,d}\, q^d.$$

They constructed another Calabi-Yau three-fold $\hat{X}$ (called a mirror of X), studied the moduli space of complex structures on $\hat{X}$, computed some period integrals, and used a string duality to predict/conclude an explicit formula for $F_0(q)$. The first few terms of F_0 corresponding to the number of rational curves on X of small degrees agree with the computations by mathematicians.

Physicists computed enumerative geometry on X via a non-enumerative computation on the mirror manifold $\hat{X}$, which is called computations in B-model or on B-side. Mathematicians still want to do enumerative computations on X, which is called computations in A-model or on A-side. In order to achieve that, mathematicians have to make several changes to the original enumerative geometric problem.

Recall that the conjecture of Clemens is yet to be proved. We don't know whether the number of rational curves on a general quintic is finite or not. Furthermore, the moduli space of curves on X is hard to work with.

Instead of counting number of curves $\mathcal{C}$ on X, we can count the number of maps from curves to X:

$$f\colon \mathcal{C} \to X.$$

If the target of the map $f\colon \mathcal{C} \to X$ is rigid (no deformation) in X and the map is an isomorphism, then the counting the number of such maps f is the same as the counting of the number of curves $\mathcal{C}$.

The set of such maps are called the moduli space of stable maps to X due to Kontsevich(1995), denoted by

$$\mathcal{M}_g(X, d),$$

where g is the genus of the curve $\mathcal{C}$, and d is the degree of the image curve $f(\mathcal{C})$ in $\mathbb{P}^4$.

An element in $\mathcal{M}_g(X, d)$ is represented by a homomorphic map f from a compact nonsingular curve $\mathcal{C}$ to X. Unfortunately, such a space $\mathcal{M}_g(X, d)$ is not compact, and hence is not a good space to perform intersection theory. Kontsevich introduced the concept of stable maps from nodal curves to X. The domain of the map f may not be a nonsingular curve, but it is at worst a union of smooth curves intersecting transversally or a singular curve with only nodal singularities. By adding these maps to $\mathcal{M}_g(X, d)$, we obtain a compactified moduli space of stable maps $\overline{\mathcal{M}}_g(X, d)$. This is the first major change in enumerating curves on X.

Naively, the dimension of $\overline{\mathcal{M}}_g(X, d)$ is expected to be zero. Rigorously, by deformation theory, the expected dimension of $\overline{\mathcal{M}}_g(X, d)$ is zero. However, the actual dimension of $\overline{\mathcal{M}}_g(X, d)$ is not zero. It is due to the similar phenomenon mentioned in the problem of counting number of conics tangent to 5 lines in $\mathbb{P}^2$, i. e. , excess intersection.

How do we overcome the difficulty of excess intersection?

In intersection theory, most of the problems of excess intersections can be reduced to computations of Chern classes of some vector bundles. Therefore, it is important to find the vector bundles.

The moduli space is equipped with the so-called deformation and obstruction theory. Due to the work of Behrend et al. (1997) and Li et al. (1998), a method of perfect obstruction theory is developed. Instead of a vector bundle in the case of excess intersection, there should be a two term complex of vector bundles. A virtual cycle $\overline{\mathcal{M}}_g(X, d)^{virt}$ can be constructed from this complex of vector bundles. If the two term complex of vector bundles has only one nonzero term which is a vector bundle, then the virtual cycle is the top Chern class of the vector bundle. Hence the virtual cycle technique is a generalization of the traditional method in intersection theory. The virtual cycle has dimension zero and sits in the moduli space $\overline{\mathcal{M}}_g(X, d)$. The Gromov-Witten invariants is defined to be (Behrend, 1997;Li et al. ,1998)

$$N_{g,d} = \deg \overline{\mathcal{M}}_g(X, d)^{virt}.$$

The number defined is actually a virtual counting. Also it is sometimes only a rational number, not an integer. The reason for being a rational number is that the moduli space should be replaced by the moduli stack. This is due to the fact that a stable map may admit nontrivial automorphisms.

The Gromov-Witten invariants have many rich inner structures (Kontsevich et al. ,1994; Ruan et al. ,1995; Kontsevich et al. ,1996; Behrend, 1997; Li et al. ,1998). It leads to new topics such as quantum cohomology and Frobenius manifolds.

The perfect obstruction theory and the concept of virtual cycles are the other important developments out of the study of counting curves on Calabi-Yau quintics. It becomes an essential part of defining various invariants for moduli problems such as Donaldson-Thomas invariants and its cousins.

How to compute the Gromov-Witten invariants just defined?

$\mathbb{P}^5$ admits a natural torus $\mathbb{C}^*$ action. For the case of genus zero, the moduli space $\overline{\mathcal{M}}_0(X, d)$ is smooth, and the virtual cycle $\overline{\mathcal{M}}_0(X, d)^{virt}$ is related to Chern classes of some vector bundles over it. Atiyah and Bott developed a theory of torus localisation, which can be used to compute Chern classes.

However, our setup is virtual, the localization formula for computations in traditional intersection theory doesn't apply directly to Gromov-Witten theory. Graber et al. (1999) developed a virtual torus localization method to deal with virtual cycles. This is another new development from the study of Gromov-Witten theory. It becomes one of the mostly used methods for computations of various invariants such Gromov-Witten invariants and Donaldson-Thomas invariants.

There is a standard recipe to carry out torus localization. However, it gives a vast amount of combinatorial data. The formula obtained by Candelas et al is an explicit formula. One of the most challenging problems is how to package these data from localization into a neat formula.

The physical method is the computation of some period integrals from the variation of Hodge structures of the mirror manifold $\hat{X}$. There is a Gauss-Manin connection, and the physical generating function $F_0^B(t)$ is a solution of a differential equation called Picard-Fuchs equation.

The breakthrough came from the work of Givental(1996;1999), Lian et al. (1999) and Bertram(2000). They carried out the computation of $F_0(q)$ by introducing an auxiliary graph space $X \times \mathbb{P}^1$, i. e. , a map $f: \mathbb{P}^1 \to X$ induces a map $g: \mathbb{P}^1 \to X \times \mathbb{P}^1$. Using the torus action on $\mathbb{P}^1$ and the ambient space $\mathbb{P}^4$, Givental studied the $\mathbb{C}^*$-equivariant Gromov-Witten invariants. He defined a Givental connection coming from the quantum product on the cohomology groups of X defined via Gromov-Witten invariants. The connection is flat due to WDVV equation, which is a property of Gromov-Witten invariants first found by physicists Witten-Dijkgraaf-Verlinde-Verlinde. Givental was able to show that $F_0^A(q)$ equals the physical generating function $F^B(t)$ via a change of variable $q = \mathrm{e}^t$.

The method developed by Givental is very sophisticated and becomes the standard method for other target manifolds which are not Calabi-Yau manifolds, and for higher genus Gromov-Witten invariants.

It is remarkable that the enumerative geometry of counting rational curves on a quintic Calabi-Yau threefold morphs into a subject of its own. Along the way, several new concepts and new methods are developed. It even ventures into different seemly unrelated research areas such as period integrals, connections, Picard-Fuchs differential equations.

How about higher genus Gromov-Witten invariants of the quintic X?

Once again, physicists Bershadsky et al. (1993) studied higher genus Gromov-Witten invariants of the Calabi-Yau quintics and obtained several surprising results about the structure of the generating function $F_g^B(t)$. In particular, they found a complete formula for genus 1 Gromov-Witten invariants $F_1(q)$. For a complete formula for higher genus Gromov-WItten invariants, one can study the paper by physicists Huang et al. (2009).

For higher genus g, the moduli space $\overline{\mathcal{M}}_g(X, d)$ is no longer smooth, and the virtual cycle is no longer a Chern class of some vector bundle. New methods are needed.

For the genus 1 case, the moduli space $\overline{\mathcal{M}}_1(X, d)$ has many irreducible components, one of which, called ghost component, consists of stable maps mapping the irreducible component with genus 1 of the reducible source curve to a point.

Vakil et al. (2008) and Li et al. (2009) analysed the moduli space $\overline{\mathcal{M}}_1(X,d)$, and they performed blowups to deal with the singularities. A formula relating the Gromov-Witten invariants with a refined invariant is obtained. Zinger(2008) worked out the computation of the refined invariants, thus proved the formula of BCOV for genus 1 case.

There are attempts to use the similar method to study other higher genus cases. However, the singularities of the moduli space for genus bigger than 1 are too complicated and prevent people to get a workable setup.

3 Reformulation of GW invariants and its cousin FJRW invariants

A stable map to the quintic $X = \{ w: = x_1^5 + \cdots + x_5^5 = 0 \}$ is described by

$$f: \mathcal{C} \to X.$$

It is also a map to $\mathbb{P}^4$. The projective space $\mathbb{P}^4$ is a geometric invariant quotient in the stack $[\mathbb{C}^5/\mathbb{C}^*]$.

Inspired by physicists Guffin et al. (2009), Chang et al. (2012) developed the theory of P-fields to study Gromov-Witten invariants. Here is a brief description.

Consider a torus $\mathbb{C}^*$ action on $\mathbb{C}^6 = \mathbb{C}^5 \times \mathbb{C}$ given by

$$t\cdot(x_1,\cdots,x_5,p)=(tx_1,\cdots,tx_5,t^{-5}p).$$

The stack $[\mathbb{C}^6/\mathbb{C}^*]$ has a GIT quotient

$$K_{\mathbb{P}^4}=\left(\mathbb{C}^6-\left\{(0,\cdots,0,p)\middle|\text{all } p\right\}\right)/\mathbb{C}^*, \tag{1}$$

which is the canonical line bundle of $\mathbb{P}^4$. It admits a function on $K_{\mathbb{P}^4}$:

$$(x_1^5+\cdots+x_5^5)p: K_{\mathbb{P}^4}\to\mathbb{C},$$

since the function is invariant under the $\mathbb{C}^*$-action on $\mathbb{C}^6$.

If we consider a map f from a curve $\mathcal{C}$ to $K_{\mathbb{P}^4}$, which is a GIT quotient, the map f can be described by

$$\left\{\mathcal{C},\mathcal{L},(\varphi_1,\cdots,\varphi_5)\in H^0(\mathcal{C},\mathcal{L})^{\oplus 5},\ \rho\in H^0(\mathcal{C},\mathcal{L}^{-5})\ \middle|(\varphi_1,\cdots,\varphi_5)\not\equiv 0\right\},$$

where $\mathcal{L}$ is the line bundle associated to the map f. The sections φ_i correspond to the coordinates x_i, and the section ρ corresponds to the coordinate p. The symbol $\not\equiv 0$ means nowhere vanishing.

The P-fields theory studies a map f from a curve $\mathcal{C}$ to $K_{\mathbb{P}^4}$ with the section corresponding to p replaced by a section $\rho\in H^0(\mathcal{C},\mathcal{L}^{-5}\otimes\omega_{\mathcal{C}})$, i. e. ,

$$\xi=\{\mathcal{C},\mathcal{L},(\varphi_1,\cdots,\varphi_5)\in H^0(\mathcal{C},\mathcal{L})^{\oplus 5},\rho\in H^0(\mathcal{C},\mathcal{L}^{-5}\omega_{\mathcal{C}})\mid(\varphi_1,\cdots,\varphi_5)\not\equiv 0\}. \tag{2}$$

The section ρ is called a P-field of the theory. We can fix some numerical data. The curve $\mathcal{C}$ has genus g, and the line bundle $\mathcal{L}$ has degree d.

The set of such ξ forms the moduli space (stack) of the theory. The moduli space is not compact due the appearance of sections ρ, and hence there is no way one can define invariants. However, the obstruction sheaf $\mathcal{O}b$ of the theory admits a co-section, i. e. ,

$$\sigma: \mathcal{O}b\to\mathcal{O}.$$

Kiem et al. (2013) developed a theory of co-section localization studying the virtual cycle defined on the degenerate locus of the co-section, which is the zero locus $\sigma^{-1}(0)$.

Applying the co-section localization to our setup, the co-section is derived from the function $w=x_1^5+\cdots+x_5^5$. The degenerate locus is $\{\rho=0,\varphi_1^5+\cdots+\varphi_5^5=0\}$, which is exactly the moduli stack $\overline{\mathcal{M}}_g(X,d)$. The co-section localization theory defines a virtual cycle lying in the degenerate locus $\overline{\mathcal{M}}_g(X,d)$, and Gromov-Witten theory also has a virtual cycle. The key result of Kiem et al. (2013) is that the two virtual cycles are equal up to a sign.

This gives a reformulation of GW invariants. The original definition of GW invariants is counting maps from curves to X, which is closed to counting curves on X. This version of Gromov-Witten theory deviates from curve counting due to the appearance of extra P-fields. P-fields theory also doesn't provide a better way to compute GW invariants. Its importance will appear when we consider a cousin of Gromov-Witten theory.

In physics, there is another theory, called Landau-Ginzburg theory, which is equivalent to Gromov-Witten theory physically. Gromov-Witten theory of the quintic $X=\{w=0\}$ studies GW invariants on the quintic X. The corresponding Landau-Ginzburg theory studies the singularity of the function w. The counterpart of GW invariants is the FJRW invariant, defined by Witten(1993) in physics and Fan et al. (2013) in mathematics. The key ingredient is to define invariants from the moduli space of spin curves. Witten enlarged the moduli space of spin curves by adding sections of the line bundle from the spin structure. However the new moduli space is not compact and cannot be used to define invariants. Witten introduced the Witten equation so that the moduli space of solutions of the Witten equation is again compact, hence invariants can be defined, called Witten top Chern class. Unfortunately, the Witten equation involves taking the complex conjugate, and hence it cannot be translated into algebraic geometric language directly.

Polishchuk et al. (2001) defined an algebraic geometric version of the invariants. Chiodo found another definition of Witten top Chern class using K-theory. Fan, Jarvis and Ruan used analytic tools to define invariants for a very general setup.

In Chang et al. (2015), there is another approach to reformulate the FJRW invariants via the method of P-fields already used for GW invariants. The moduli space (stack) $\overline{\mathcal{M}}_g^{1/5,5p}$ of the theory consists of the following objects

$$\xi = \left\{\mathcal{C}, \mathcal{L}, \left(\varphi_1, \cdots, \varphi_5\right) \in H^0\left(\mathcal{C}, \mathcal{L}\right)^{\oplus 5}, \rho \in H^0\left(\mathcal{C}, \mathcal{L}^{-5}\omega_{\mathcal{C}}\right) | \rho \not\equiv 0\right\}. \tag{3}$$

Since the section ρ is nowhere vanishing, it implies that $\mathcal{L}$ is a 5-spin line bundle. Hence it requires the underlying curve $\mathcal{C}$ be a twisted curve (orbi-curve) (Abramovich et al. ,2003) and $\mathcal{L}$ be an orbi-line-bundle. Sections $\varphi_1,\cdots,\varphi_5$ are the P-fields of the theory.

One obtains (3) by considering a map from a (twisted) curve $\mathcal{C}$ to the orbifold $\left[\mathbb{C}^5/\boldsymbol{\mu}_5\right]$ where $\boldsymbol{\mu}_5$ is the cyclic group of roots of fifth unity acting on $\mathbb{C}^5$ diagonally. It can also be viewed as a map from a curve $\mathcal{C}$ to the following GIT quotient:

$$\left(\mathbb{C}^6 - \left\{\left(x_1, \cdots, x_5, 0\right) \middle| \text{all } x_1, \cdots, x_5\right\}\right)/\mathbb{C}^*, \tag{4}$$

with an $\omega_{\mathcal{C}}$ twist on the section ρ similar to P-fields formulation of GW invariants.

Clearly the moduli space $\overline{\mathcal{M}}_g^{1/5,5p}$ is not compact. Again, the function $w = x_1^5 + \cdots + x^5$ induces a co-section of the obstruction sheaf of the theory. By the co-section localization method, the degenerate locus of the co-section is the moduli space of 5-spin twisted curves. The localized virtual cycle defines FJRW invariants in the narrow sector. It is also proved that all the existing definitions of Witten top Chern class agree.

Using physical arguments, the generating function of GW invariants of the quintic, after some change of variables and some transformations, equals the generating function of FJRW invariants of the quintic polynomial. This is called CY-LG correspondence.

4 Mixed-spin-P-fields and its variants

In the reformulations of both GW invariants and FJRW invariants, we can see the same description in (2) and (3). The only difference is that $\left(\varphi_1,\cdots,\varphi_5\right)$ is nowhere zero for GW invariants and ρ is nowhere zero for FJRW invariants. It suggests a deeper relation between these two invariants.

We can compare the GIT quotients (1) and (4). They are the GIT quotients of two different stability conditions of the stack $\left[\mathbb{C}^6/\mathbb{C}^*\right]$. One can consider the variations of stability conditions. To be precise, we can construct the following space:

$$Z = \left(\mathbb{C}^6 \times \mathbb{P}^1 - \left\{\left(0, \cdots, 0, p\right) \times \left[0, 1\right]\right\} \cup \left\{\left(x_1, \cdots, x_5, 0\right) \times \left[1, 0\right]\right\}\right)/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by weights $\left(1,1,1,1,1,-5,1,0\right)$.

Let $\left[v_1,v_2\right]$ be the coordinates of the projective line $\mathbb{P}^1$. When $v_2 = 0$, it is $\mathbb{C}^5/\boldsymbol{\mu}_5$; when $v_1 = 0$, it is $K_{\mathbb{P}^4}$. Hence Z is the path connecting two GIT quotients of $\left[\mathbb{C}^6/\mathbb{C}^*\right]$.

If we consider maps from a (twisted) curve $\mathcal{C}$ to Z (also with an $\omega_{\mathcal{C}}$ twist on sections ρ), we get the following equivalent description,

$$\begin{aligned} \xi \quad = \quad & \left(\mathcal{C}, \mathcal{L}, \mathcal{N}, \left(\varphi_1, \cdots, \varphi_5\right) \in H^0\left(\mathcal{L}^{\oplus 5}\right), \rho \in H^0\left(\mathcal{L}^{\vee 5}\omega_{\mathcal{C}}\right), \nu_1 \in H^0\left(\mathcal{L} \otimes \mathcal{N}\right),\right. \\ & \left. \nu_2 \in H^0\left(\mathcal{N}\right) \middle| \left(\varphi_1, \cdots, \varphi_5, \nu_1\right), \left(\rho, \nu_2\right), \left(\nu_1, \nu_2\right) \text{all nowhere zero}\right). \end{aligned}$$

Such a collection is called a mixed-spin-P-field (MSP for short).

If $\nu_1 = 0$, $\left(\varphi_1,\cdots,\varphi_5\right)$ nowhere zero and ν_2 nowhere zero. Thus we have $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$. We get the GW-invariants via Chang-J. Li's reformulation.

If $\nu_2 = 0$, ρ nowhere zero and ν_1 nowhere zero. Thus we get $\mathcal{N} \cong \mathcal{L}^{\vee}$ and $\mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}$. We get FJRW invariants via Chang-J. Li-Li's definition.

If $\rho = 0$ and $(\varphi_1,\cdots,\varphi_5) = \vec{0}$, then ν_1 and ν_2 are nowhere zero. We have $\mathcal{L} \cong \mathcal{O}$ and $\mathcal{N} \cong \mathcal{O}$. We get the moduli space of curves.

We can fix numerical invariants: the genus g of the curve $\mathcal{C}$, $d_0 = \deg(\mathcal{L} \otimes \mathcal{N})$, $d_\infty = \deg \mathcal{N}$. Let $\mathfrak{W}$ be the moduli stack of such ξ with a fixed set of numerical invariants. The virtual dimension of the moduli stack is $\delta = d_0 + d_\infty + 1 - g$.

The moduli space (stack) of such objects is not compact. Again, its obstruction sheaf admits a co-section whose degenerate locus can be proved to be compact (Chang et al. ,2019). Therefore, there is a compact virtual cycle $[\mathfrak{W}]^{vir}_{loc}$.

We consider a torus $T = \mathbb{C}^*$ action on the moduli stack via scaling the section ν_1 only. We don't use the moduli space to construction invariants. Instead, making use of the torus action, we can perform computations on the moduli stack. In fact, we can derive a class of vanishings using the virtual cycle. Applying the virtual localization formula to these vanishings, we can obtain polynomial relations among the GW invariants and the FJRW invariants of Fermat quintic polynomials:

$$\sum_{\Gamma} \mathrm{res}_{\mathrm{t}=0}\left(\mathrm{t}^{\delta-1} \cdot \frac{[\mathfrak{W}^T_\Gamma]^{vir}_{loc}}{e(N_{\mathfrak{W}^T_\Gamma/\mathfrak{W}})}\right)_0 = 0, \text{ when } \delta > 0. \tag{5}$$

Here Γ is a graph coming out of torus localization, and t is the equivariant parameter for the group T.

To see if the relations (5) give an effective method to compute the GW invariants of the quintic,Chang et al. (2020) worked on genus 1 GW invariants, which was computed earlier by Zinger. Using the combinatoric techniques in (Zinger, 2008), and some other methods to get rid of combinatoric difficulties, we reproved Zinger's formula. The upshot in this work is that we didn't do analysis such as blowups on the moduli stack of stable curves nor on the moduli stack of mixed-spin-P-fields. The reason is that, by adding LG sector to the moduli space of CY sector, LG sector plays the role of resolution of singularities caused by the ghost component. Clearly, this cannot be done if we use the moduli space of stable maps. We used the P-fields formulation of GW invariants and FJRW invariants, which gives a natural platform to combine them in one geometric setup, i. e. , mixed-spin-P-fields.

MSP theory is also used to compute genus 1 FJRW invariants by Guo et al. (2019). These results demonstrate that MSP theory provides an effective method to calculate GW invariants and FJRW invariants. Geometry part of MSP theory has been all established by Chang et al. (2019; 2020; 2022). The most essential remaining part is to find a workable method to package the complicated combinatoric data.

Using MSP theory to compute genus 2 GW invariants met some combinatoric difficulties. It is discovered that if we modify our setup by enlarge the number of v_1 fields to N so that the MSP theory becomes N-MSP theory, and modify Givental's R-matrix method, N-MSP theory can provide an effective tool to handle all genus GW invariants of quintics. The object in N-MSP theory is given by

$$\begin{aligned}\xi \ = \ \Big(&\mathcal{C}, \mathcal{L}, \mathcal{N}, (\varphi_1,\cdots,\varphi_5) \in H^0(\mathcal{L}^{\oplus 5}),\ \rho \in H^0(\mathcal{L}^{\vee 5}\omega_{\mathcal{C}}),\ \mu \in H^0(\mathcal{L} \otimes \mathcal{N})^{\oplus N},\\ &\nu \in H^0(\mathcal{N}) \Big| (\varphi_1,\cdots,\varphi_5,\mu),\ (\rho,\nu),\ (\mu,\nu) \text{ all nowhere zero}\Big).\end{aligned}$$

A miraculous effect is that counting of chains of rational curves in these N-MSP theory gives precise formula for the *propagator* physicists obtained by solving differential equations in B-model theory, which plays a pivotal role in B-side higher genus theory. After discovering this fact, in the sequence of papers (Chang et al. ,2018; Chang et al. ,2021a;Chang et al. ,2021b), several key conjectures in higher genus GW invariants of quintics, such as Yamaguchi-Yau conjecture (Yamaguchi et al. ,2004) and Bershadsky-Cecotti-Ooguri-Vafa conjecture (Bershadsky et al. ,1993) , are all proved. Regarding its potential applicability to general CY threefolds, the N-MSP theory is now expected to provide the correct framework as the counterpart of the fruitful B-model physical structure on A-side.

There are other works on higher genus GW invariants such as a series of papers by Guo et al. (2018), Lho et al. (2018) and Chen et al. (2021).

In summary, the counting of curves on Calabi-Yau quintics is a very classical problem in enumerative geometry. Yet, due to the infusion of works by physicists and their visions, it developed into one of the hottest research areas for the last thirty years. Along the way, many new concepts and methods are developed. They are not only used to solve curve counting problems on quintics, but also for other enumerative geometry problems arising from moduli spaces of various objects such as bundles on Calabi-Yau threefolds.

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